Combinatorial Game Theory

Mathematical Sciences Dept.

February 28, 2020

0. Preamble

1. Nim and Definitions

2. Sprague-Grundy Numbers

3. Laskar’s Nim and Split-q-Nim

4. The Caterpillar Game

5. Nim with a Pass

6. Amalgamation Nim

Preamble. The speaker has always been interested in Combinatorial Game Theory, but has only recently stated publishing in this area.

We start with an old game, Nim. This has been completely solved (Bouton, 1902). More importantly, it is the basic for playing more general games, using Sprague-Grundy numbering of the possible game positions (1935, 1939). After discussing these two topics, we move on to another old game, Laskar's Nim (1931), and Split-q-Nim (2018).

The Caterpillar Game was introduced by Harary (2001), and the speaker listed some results on a web page for a Recreational Mathematics course (2003). Recently, those results have been expanded to cover 95% of the starting positions (submitted 2019).

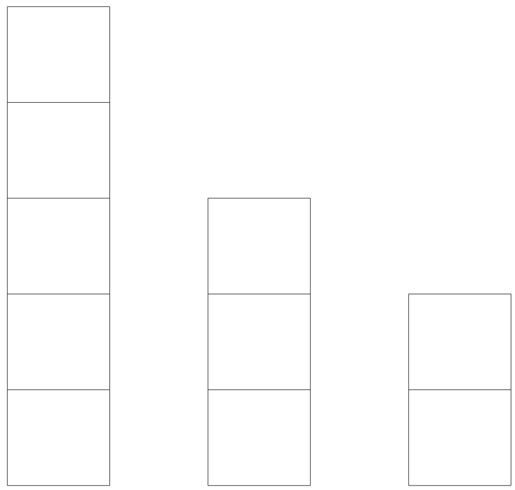
Nim with a Pass is a version of Nim in which one of the players may pass instead of taking a normal move. Once any player has passed, the other player may not pass during that game. The addition of this extra move complicates the analysis of the game.

Amalgamation Nim was suggested by a student (Dawne Richards) after the speaker had discussed Laskar's Nim in class in 2001. Paul Yiu posted this problem on his problems page. In 2019, Ben Handley contacted me to say that the problem seemed non-trivial, and we submitted a paper (currently under revision).

1. Definitions, and the Game of Nim.

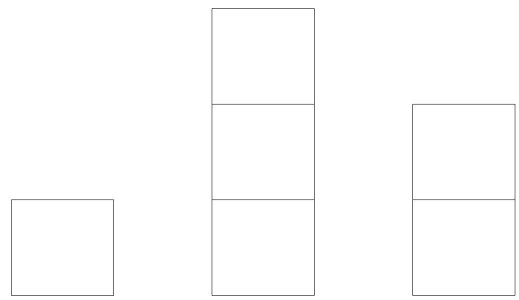
Let’s start with the Game of Nim, and talk about definitions later. On the table there are several piles of coins. There are two players, who will alternate moves. At each player’s turn, the player removes some positive number of coins from any one of the piles. The player who moves last wins.

To make the first example easy, let’s suppose that there are three piles of coins, with 5, 3, and 2 coins in the piles.

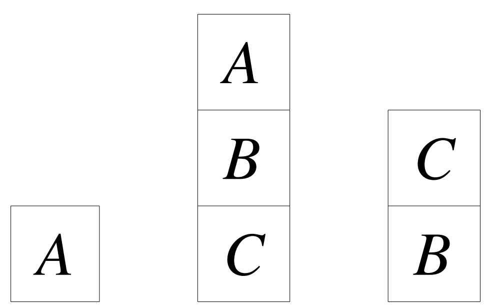


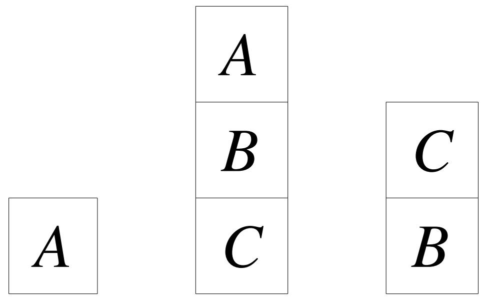
What should the next player do? Hint: There is exactly one winning move.

If I remove four coins from the first pile, I leave you with



and it is easy to see that there are no winning moves from the position [1,3,2]. An easy way to see this is with the following picture:





If you take one of the coins labeled A, I will take the other coin labeled A, leaving position [0,2,2]. If you take one of the coins labeled B, I will take the other B, leaving position [1,1,0]. If you take one of the C coins, I will take the other C, leaving position [1,0,1]. In each case, when a player takes a coin, the player takes all of the coins above that coin as well.

The game is easy to win if you leave your opponent with an even number of piles of each height. You play a mirror strategy.

Now, suppose there are five piles, and the piles have 12, 13, 14, 16, and 31 coins. If it is your turn to play, what would be your move? It would be somewhat more difficult to give an ad hoc argument. This is where it is useful to know Bouton’s strategy.

If a and b are non-negative integers, we define $a \oplus b$ to be the Nim-sum or bit-sum of a and b. If

$$a = \sum_{k=0}^{N} a_k 2^k \qquad and \qquad b = \sum_{k=0}^{N} b_k 2^k,$$

with $a_k, b_k \in \{0, 1\}$ for each k, then

$$a \oplus b = \sum_{k=0}^{N} (a_k \oplus b_k) 2^k,$$

where $\quad 0 \oplus 0 = 0 = 1 \oplus 1,$
$\quad 0 \oplus 1 = 1 = 1 \oplus 0.$

This operation is associative and commutative.

For position $[12, 13, 14, 16, 31]$, we calculate

$$\begin{aligned} & 12 \oplus 13 \oplus 14 \oplus 16 \oplus 31 \\ = \ & 1 \oplus 14 \oplus 16 \oplus 31 \\ = \ & 15 \oplus 16 \oplus 31 \\ = \ & 31 \oplus 31 \\ = \ & 0 \end{aligned}$$

Any move from this position would result in a position with a non-zero Nim-sum.

Conversely, from any position $[x_1, x_2, \ldots, x_m]$ with $x_1 \oplus x_2 \oplus \cdots \oplus x_m = y \neq 0$, there is a move to a position with Nim-sum zero. There will be some pile, x_j such that $x'_j = x_j \oplus y < x_j$. A winning move is to reduce a pile of height x_j to height x'_j.

Some definitions.

We deal with two-person normal, impartial, combinatorial games. There is no element of chance in any of the moves. These games are sometimes called “Games of No Chance”. A game is determined by a set of positions, an initial position, and a set of moves from each position. That is, a combinatorial game can be modelled as a directed graph.

Normal: the player who makes the last move wins.

Impartial: both players would have the same move from any given position, if it were that player’s turn to move. Compare to Chess, where one player moves White pieces and the other player moves Black pieces.

We ban cycles from our games: that is, there can be no repeated position, and the graph representing the game is a directed, acyclic graph. We also restrict our attention to finite games.

2. Sprague-Grundy Numbers

For a two-person normal, impartial, combinatorial game, with no repeated positions, we can define a function as follows.

A position is terminal if there are no moves from that position. Any terminal position, X, will receive the Sprague-Grundy number zero. We write $\text{Gr}(X) = 0$. Also, we will usually refer to Sprague-Grundy numbers simply as Grundy numbers.

In general, let X be any position, and let Y be the set of positions that can be reached from X in a single move. If all positions in Y have already received Grundy numbers, then $\text{Gr}(X) = \text{mex}\left\{\text{Gr}(K) : K \in Y\right\}$.

The function mex is defined by:
mex$(S) = \min\{v \in \mathbf{N} : v \notin S\}$, where $\mathbf{N}$ is the set of non-negative integers. Thus, mex$\{2, 0, 5, 1, 8\} = 3$.

Grundy numbers are extremely useful if the two players are simultaneously playing two or more normal games at the same time, with each player at his or her turn selecting one of the games to play in and then a move to make in that game. The next player may choose to play in a different game, of course.

The analysis of the collection of games is easy. Calculate the Grundy number of each of the subgames, and play Nim on the Grundy numbers. If the Nim move is to reduce game$_k$ from Grundy number q to Grundy number s, $s < q$, then there must be a move in that game to a position with Grundy number s because of the definition of the mex function and the Grundy labelling.

We give a small example in the next game.

3. Laskar's Nim and Split-q-Nim

Laskar's Nim is played like Nim, with one additional move available: a pile of height two or more may be split into two piles. The player who takes the last coin wins.

For a game with one pile of height n, it is easy enough to calculate

n	Gr(n)
0	0
1	1
2	2

When we come to calculate Gr(3), we have the possible moves: $3 \to 0$, $3 \to 1$, $3 \to 2$, and $3 \to [1,2]$. Here, $[1,2]$ stands for the game with two subgames, one of which is a pile of height 1 and the other a pile of height 2. Note that Gr$([1,2])$ = Gr$(1) \oplus$ Gr$(2) = 1 \oplus 2 = 3$.

Hence,

$$\begin{aligned} Gr(3) &= \text{mex}\Big\{\text{Gr}(0), \text{Gr}(1), \text{Gr}(2), \text{Gr}([1,2])\Big\} \\ &= \text{mex}\{0,1,2,3\} = 4. \end{aligned}$$

Similarly,

$$\begin{aligned} Gr(4) &= \text{mex}\Big\{\text{Gr}(0), \text{Gr}(1), \text{Gr}(2), \text{Gr}(3), \\ &\qquad \text{Gr}([1,3]), \text{Gr}([2,2])\Big\} \\ &= \text{mex}\{0,1,2,4,1 \oplus 4, 2 \oplus 2\} \\ &= \text{mex}\{0,1,2,4,5,0\} = 3. \end{aligned}$$

It is easy enough to continue this process to show that

$$\begin{aligned} \text{Gr}(0) &= 0, \\ \text{Gr}(4k+1) &= 1, \\ \text{Gr}(4k+2) &= 2, \\ \text{Gr}(4k+3) &= 4, \\ \text{Gr}(4k+4) &= 3, \end{aligned}$$

for every non-negative integer k.

Laskar's Nim for a small game: $[2, 4, 6]$.

First, the Grundy numbers are $[2, 3, 6]$ and $2 \oplus 3 \oplus 6 = 1 \oplus 6 = 7$. We calculate $2 \oplus 7 = 5$, $3 \oplus 7 = 4$, $6 \oplus 7 = 1$.

Replacing the pile of height six by a position with Grundy number 1 must be possible by the definition of the Grundy function.

It also happens, here, that the pile of height 4, with Grundy number 3, can be replaced by a pile of height 3, with Grundy number 4. This move is not one that is guaranteed to occur from the definition of the Grundy function.

It is not possible to replace the pile of height 2 by any other position to attain a Grundy number of 5 for that position.

The possible winning moves are:
$[2,4,6] \to [2,3,6]$, $[2,4,6] \to [2,4,1]$, and $[2,4,6] \to [2,4,2,4]$.

Split-q-**Nim**.

As I walked to class (MAT 4937)in 2017, I wanted to give my students a 3-week project. If I gave them Laskar's game, they'd be able to find it on the web. So, I modified it a bit.

At a player's turn, the player may make a regular Nim-move or the player could pick a pile, add 2 coins to the pile, and then split the new pile into two piles, both of which are shorter than the original pile. We call this game Split-2-Nim.

It is easy enough to see that the game must end, and that no position is repeated, if we order the positions lexicographically by the heights of the piles.

So, what’s the winning strategy?

The students had three weeks, you have a few seconds.

We calculate the Grundy numbers for a pile of height n for a few small values of n.

n	$\mathrm{Gr}(n)$
0	0
1	1
2	2
3	3
4	4
5	5

Nothing surprising yet. The moves from a pile of height 5 are $5 \to 0$, $5 \to 1$, $5 \to 2$, $5 \to 3$, $5 \to 4$, $5 \to [3,4]$. So,

$$\begin{aligned}\text{Gr}(5) &= \text{mex}\Big\{\text{Gr}(0), \text{Gr}(1), \text{Gr}(2), \text{Gr}(3), \text{Gr}(4), \\ &\qquad \text{Gr}([3,4])\Big\} \\ &= \text{mex}\{0,1,2,3,4,3 \oplus 4\} \\ &= \text{mex}\{0,1,2,3,4,7\} = 5.\end{aligned}$$

The moves from a pile of height 6 are $6 \to 0$, $6 \to 1$, $6 \to 2$, $6 \to 3$, $6 \to 4$, $6 \to 5$, $6 \to [4,4]$, $6 \to [3,5]$. So,

$$\begin{aligned}\text{Gr}(6) &= \text{mex}\Big\{\text{Gr}(0), \text{Gr}(1), \text{Gr}(2), \text{Gr}(3), \text{Gr}(4), \\ &\qquad \text{Gr}(5), \text{Gr}([4,4]), \text{Gr}([3,5])\Big\} \\ &= \text{mex}\{0,1,2,3,4,5,4 \oplus 4, 3 \oplus 5\} \\ &= \text{mex}\{0,1,2,3,4,5,0,6\} = 7.\end{aligned}$$

The students were shown this much when the problem was assigned. Did you guess the pattern?

I came back to them a week later with the pattern, but they didn't even record this correctly.

It turns out that $\text{Gr}(n) \in \{n, n \oplus 1\}$.

Let $f(n)$ denote the number of “1” bits in the binary representation of $\left\lfloor \frac{n}{2} \right\rfloor$.

Let $g(n) = 1$ if $f(n)$ is even and let $g(n) = 0$ if $f(n)$ is odd.

Then, for $n > 1$, Gr$(n) = n \oplus g(n)$. We will skip the proof.

Note: the sequence $g(0), g(2), g(4), \ldots$ is a Thue-Morse sequence.

We generalize the problem: In Split-q-Nim, a player can make a normal Nim-move, or may select a pile, add q coins to the pile, and split the pile into two piles smaller than the original pile.

If $q < 0$, or if q is odd, Gr$(n) = n$.

If $q = 0$, we have Laskar's Nim and Gr$(n) \in \{n, n \oplus 1\}$ depending on the last two bits of n.

If $q = 2$, we have Split-2-Nim, discussed already.

What about q even, $q > 2$? If $2^b \leqslant q < 2^{b+1}$, then Gr$(n) \in \{n, n \oplus 1\}$, generally depending on the last b bits of n. There is a special case when n is close to a power of 2. If you want the exact calculation, look at the paper. We give only a brief glimpse here:

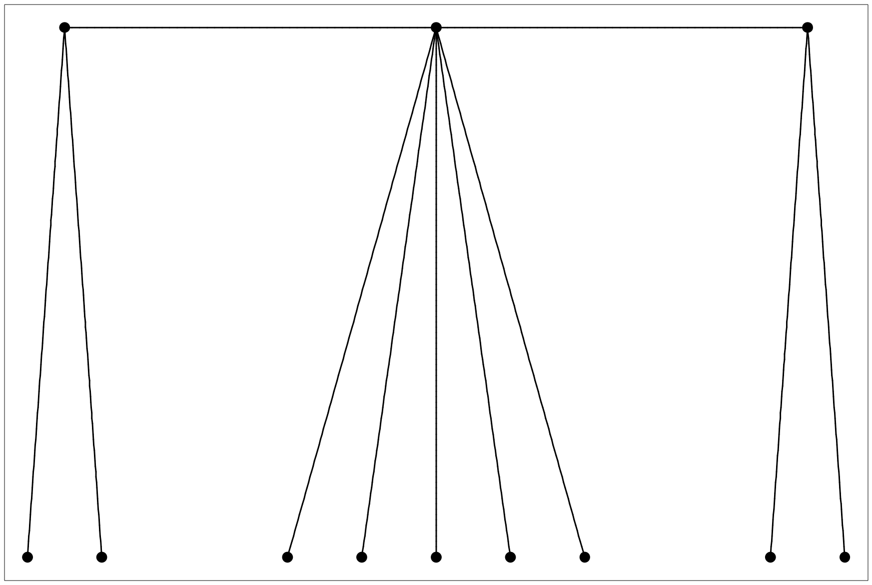
Theorem. Suppose that $q = 2^k + 4r$ or $q = 2^k + 4r + 2$, $0 \leqslant r \leqslant 2^{k-2} - 1$, and that $n = 2^k m + j$, $0 \leqslant j \leqslant 2^k - 1$. If $j \in M(2^{k-1} - 1 - 2r)$ and m is not a power of 2, then $\mathrm{Gr}(n) = n \oplus 1$, and for all other choices of j and m, $\mathrm{Gr}(n) = n$.

Here, $j \in M(2^{k-1} - 1 - 2r)$ if the bit pattern for j is dominated by the bit pattern for $2^{k-1} - 1 - 2r$. That is, in the binary representation of $2^{k-1} - 1 - 2r$, there is a 1 in every place in which there is a 1 in the binary representation of j.

Reference. Locke & Gray, Discrete Math, 2018.

4. The Caterpillar Game

A caterpillar is a connected acyclic graph (a tree), with the property that when all of the leaves (vertices of degree one) are deleted, the resulting tree is a path. We allow the possibility that the path is an empty graph. We can specify a caterpillar by listing how many legs there are at each vertex of the spine. For example, [2,5,3] would be the caterpillar with three vertices on the spine, the first spine vertex having 2 legs, the second having 5 legs, and the third with 3 legs.



Caterpillar $[2,5,3]$

The caterpillar [0] has one vertex and no edges. The caterpillar [1] has two vertices and one edge. This is the caterpillar that produces an empty graph if we delete the vertices of degree one.

Harary's caterpillar game proceeds as follows. There are two players who alternate turns. The player who makes the last move wins. At each of his turns, a player removes the edges from a path P in the remaining graph, as long as P has at least one edge.

Note that after one move, it is quite possible that the caterpillar has been reduced to a collection of smaller caterpillars. It will not necessarily be a single caterpillar. Thus, the game is really played on a collection of caterpillars.

Easy cases. The cases in which the caterpillar is a star $[n]$ are easy. The Grundy numbers are given by Gr($[3k]$) $= 0$, Gr($[3k+1]$) $= 1$, Gr($[3k+2]$) $= 2$.

Given an arbitrary caterpillar, a good partial strategy is to taken away the edges on the spine and zero or one edges incident with the first vertex on the spine and zero or one edges at the other end of the spine.

For the caterpillar [2,5,3], take only the two spine edges, resulting in [2],[5],[3], which is a P-position.

For the caterpillar [3,5,3], we would also reduce to [2],[5],[3].

For the caterpillar [1,5,3], we would reduce to [0],[5],[2].

What would you do with the caterpillar [1,5,1]?

Many years ago, I posted a web-page with Grundy numbers for caterpillars $[a,b]$. A few years ago, Richard Low asked me if I had any game theory projects going, and I mentioned that old page and gave him permission to work it up into a paper. The result is a multi-author submission to Integers, in which we show a winning first move for 95% of the possible initial caterpillars. These first moves leave stars and caterpillars $[a',b']$ with two spine vertices (bi-stars).

The Grundy number for a bi-star $[a, b]$ is at most 6. If $b \geqslant 1$, let $b = 3k + j, j \in \{0, 1, 2\}$. Then,

$$\mathrm{Gr}([1, 3k]) = 5$$
$$\mathrm{Gr}([1, 3k+1]) = 3$$
$$\mathrm{Gr}([1, 3k+2]) = 4$$

$$\mathrm{Gr}([2, 3k]) = 6$$
$$\mathrm{Gr}([2, 3k+1]) = 4$$
$$\mathrm{Gr}([2, 3k+2]) = 1$$

$$\mathrm{Gr}([3, 3k]) = 1$$
$$\mathrm{Gr}([3, 3k+1]) = 5$$
$$\mathrm{Gr}([3, 3k+2]) = 6$$

$$\mathrm{Gr}([4, 3k]) = 5$$
$$\mathrm{Gr}([4, 3k+1]) = 2, \text{ if } b \geqslant 2$$
$$\mathrm{Gr}([4, 3k+2]) = 4$$

$$\text{Gr}([5, 3k]) = 6$$
$$\text{Gr}([5, 3k+1]) = 4$$
$$\text{Gr}([5, 3k+2]) = 1$$

$$\text{Gr}([6, 3k]) = 1$$
$$\text{Gr}([6, 3k+1]) = 5$$
$$\text{Gr}([6, 3k+2]) = 6$$

When $a, b \geqslant 4$, these values cycle:

	$b = 3k$	$b = 3k+1$	$b = 3k+2$
$a = 3t$	1	5	6
$a = 3t+1$	5	2	4
$a = 3t+2$	6	4	1

The strategy is easy to implement: Remove all but the leftmost and rightmost spine edges, and then possibly a few other edges in the leftmost and rightmost bi-stars. There are sixteen possible ways to do this. You have to know the Grundy numbers for the bi-stars and stars, and you try to arrange it so that the Nim-sum is zero. You can achieve this 95% of the time.

95% sounds good, since one would assume that must be some initial caterpillars which are themselves P-positions: we've already seen infinitely many. But, there are rather few P-positions among the collection of all caterpillars.

Reference. W.H. Chan, Ardak Kapbasov, Arman Kapbasov, S.C. Locke, R. Low, *A Codex of N- and P-positions in Harary's Caterpillar Game*. Submitted to Integers, 2019.

5. Nim with a Pass

Generally, the normal forms of games are easy to solve than misère versions, in which the player to move last loses.

Misère Nim has an easy strategy: you play like normal Nim until you don’t.

Specifically, you play misère Nim like you are playing normal Nim unless your move would leave no piles with more than one coin. At this stage, normal Nim would have you leave an even number of piles of height one. Misère Nim would have you leave an odd number of piles of height one.

Nim is a “tame” game – for the reason described above: play like normal until what is left are piles of height one and one pile not of height one. Then, choose to leave the correct parity of piles for whichever version you are playing.

A way to make a version of Nim which appears to be harder to analyze is to allow that one player may, at some time during the game, opt to “pass”. That is, he play simply forgoes his move, and lets the opponent have another move. This pass cannot be taken as the last move of the game. Also, once one player takes a pass, the other does not have that option.

We’re not going to deal with this game any more than this. You can see that just handling the max height four case was sufficient for publication.

Reference. W.H. Chan, S.C. Locke, Richard M. Low, O.L. Wong. *A map of the P-positions in* ‘Nim With a Pass’ *played on heap sizes of at most four*. Discrete Applied Math., 2018.

6. Amalgamation Nim

This version of Nim arose when Ms. Richards asked a follow-up question to Laskar's (splitting) Nim. What can we say about the game in which a player may make a regular Nim-move, or may choose two of the piles and amalgamate them into one pile?

So, from [1,4,5], a play could move [1,4,5]→[5,5] or [1,4,5]→[4,6] or [1,4,5]→[1,9] in addition to the ten usual Nim-moves. Here, the player would select [1,4,5]→[5,5], since that leaves a P-position.

The two-pile game is easy to analyze. $[a, b]$ is a P-position iff $a = b$. In general, if there are $2m$ piles, and the number of piles of each height is even, this is a P-position. Either copy the opponent's move, or, if the opponent amalgamated two piles of the same height, remove that new pile.

The analysis for three piles is quite different from that for two piles. We list some P-positions $[a_1, a_2, a_3]$ with $a_1 \leqslant a_2 \leqslant a_3$, to get you started. In each list, $k \geqslant 0$.

$$[1, 4k + 2, 4k + 3],\ [1, 4k + 3, 4k + 5]$$

$$[2, 2k + 1, 2k + 4]$$

$$[3, 4, 8],\ [3, 9, 10],\ [3, 7, 11],\ [3, 2k + 12, 2k + 13]$$

$$[4, 5, 6],\ [4, 7, 12],\ [4, 9, 11],\ [4, 10 + 4k, 13 + 4k],\ [4, 15 + 4k, 16 + 4k]$$

$$[5, 7, 13],\ [5, 9, 15],\ [5, 10, 11],\ [5, 12 + 4k, 14 + 4k],\ [5, 17 + 4k, 19 + 4k]$$

$$[6, 7, 14],\ [6, 9, 13],\ [6, 10, 15],\ [6, 11, 12],\ [6, 16 + 4k, 19 + 4k],\ [6, 17 + 4k, 18 + 4k]$$

After this point, the periods increase.

For $a_1 = 7$, the period is 8.
For $a_1 = 8$, the period is 40.
For $a_1 = 9$, the period is 40.
For $a_1 = 10$, the period is 160.
For $a_1 = 11$, the period is 960.

Good luck!

Reference. S.C. Locke and B. Handley, *Amalgamation Nim*, submitted to Integers, 2019. Under revision.

General References.

M.H. Albert, R.J. Nowakowski, **Games of No Chance 3**, **Math**. **Sci**. **Res**. **Inst**. **Publ**., **vol**. **56**, Cambridge Univ. Press, Cambridge, 2009.

M.H. Albert, R.J. Nowakowski, D. Wolfe, **Lessons in Play**: **An Introduction to Combinatorial Game Theory**, A.K. Peters Ltd., Wellesley, Massachusetts, 2007.

E.R. Berlekamp, J.H. Conway, R.K. Guy, **Winning Ways for Your Mathematical Plays**, (Four Volumes), A.K. Peters Ltd., Wellesley, Massachusetts, 2001.

C.L. Bouton, *Nim, a game with a complete mathematical theory*, **Annals of Mathematics**, Vol. 3, second series, 1901-1902, pp. 35–39.

N.J. Calkin, K. James, J. Janoski, S. Leggett, B. Richards, N. Sitaraman and S. Thomas. *Computing strategies for graphical Nim*, **Congr**. **Numer**. **202** (2010), 171-185.

J.H. Conway, **On Numbers and Games**, second ed., A.K. Peters Ltd., Wellesley, Massachusetts, 2001.

L.A. Erickson. **The Game of Nim on Graphs**. Ph.D. Thesis - North Dakota

State University. 2011. 64pp.

A.S. Fraenkel, *Combinatorial games: selected bibliography with a succinct gourmet introduction*, **Electron**. **J**. **Combin**. **Dyn**. **Surv**. (2012) #DS2.

M. Fukuyama. *A Nim game played on graphs. II.*, **Theoret**. **Comput**. **Sci**. **304**(1-3), (2003), 401-419.

M. Fukuyama. *A Nim game played on graphs.*, **Theoret**. **Comput**. **Sci**. **304**(1-3), (2003), 387-399.

R.K. Guy, R.J. Nowakowski, *More Games of No Chance*, **Math**. **Sci**. **Res**. **Inst**. **Publ**., **vol**. **42**, Cambridge Univ. Press, Cambridge, 2002.

R.K. Guy, R.J. Nowakowski, *Unsolved problems in combinatorial games*, **Games of No Chance 3**, **Math**. **Sci**. **Res**. **Inst**. **Publ**., **vol**. **56**, Cambridge Univ. Press, Cambridge, 2009.

F. Harary. Lecture (35th Southeastern Conference on Combinatorics, Graph Theory and Computing), Baton Rouge, LA, March 2001.

P. Harding and P. Ottaway. *Edge deletion* *games with parity rules*, **Integers 14** (2014), #G1, 10pp. (electronic).

D.G. Horrocks, R.J. Nowakowski, *Regularity in the G-sequences of octal games with a pass*, **Integers 3** (2003) #G1, 10pp. (electronic).

R. Miyadera, M. Inoue, M. Fukui, *Impartial chocolate bar games with a pass*, **Integers 16** (2016) #G5, 26pp. (electronic).

M. Kano. *An edge-removing game on a graph (a generalization of Nim and Kayles)*, **Surikaisekikenkyusho Kokyuroku No**. **820**, (1993), 82-90.

R.M. Low and W.H. Chan. *Notes on the combinatorial game: Graph Nim*, **Electron**. **J**. **Graph Theory Appl**. **4** (2016), no. 2, 190-205.

L. Merchant. *Nim on the complete graph*, **Ars Combin**. **119** (2015), 117-127.

R.E. Morrison, E.J. Friedman, A.S. Landsberg, *Combinatorial games with a pass: a dynamical systems approach*, **Chaos 21** (4) (2011) 043108.

R.J. Nowakowski, **Games of No Chance**, **Math**. **Sci**. **Res**. **Inst**. **Publ**., **vol**. **29**, Cambridge Univ. Press, Cambridge, 1996.

R.J. Nowakowski, **Games of No Chance 4**, **Math**. **Sci**. **Res**. **Inst**. **Publ**., **vol**. **63**, Cambridge Univ. Press, Cambridge, 2015.

A.N. Siegel, **Combinatorial Game Theory**, **Graduate Studies in Mathematics**, **vol**. **146**, American Mathematical Society, Providence, RI., 2013.

Thank you.

What? You're still here?

Here's one more game we call is Bulgarian Nim.

At each turn, a player makes a Nim-move, and then, if there are any coins, the player makes a Bulgarian solitaire move.

A Bulgarian solitaire move: take one coin from each pile and use these coins to make a new pile.

Come to the conference to hear preliminary results on this game.